

Quantum double and κ -Poincaré symmetries in (2+1)-gravity and Chern-Simons theory

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Abstract

We review the role of Drinfeld doubles and κ -Poincaré symmetries in quantised (2+1)-gravity and Chern-Simons theory. We discuss the conditions under which a given Hopf algebra symmetry is compatible with a Chern-Simons theory and determine this compatibility explicitly for the Drinfeld doubles and κ -Poincaré symmetries associated with the isometry groups of (2+1)-gravity. In particular, we explain that the usual κ -Poincaré symmetries with a timelike deformation are not directly associated with (2+1)-gravity. These κ -Poincaré symmetries are linked to Chern-Simons theory only in the de Sitter case, and the relevant Chern-Simons theory is physically inequivalent to (2+1)-gravity.

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1 Introduction and Motivation

Since their discovery in [1, 2], κ -Poincaré symmetries have attracted much interest as possible symmetries of a quantum theory of gravity in three and higher dimensions. The idea is that the usual Poincaré symmetry of Minkowski space or, more generally, the isometry groups of de Sitter and anti de Sitter space, are deformed into Hopf algebras. These Hopf algebras, which have been shown to have the structure of a bicrossproduct [3], depend on a parameter κ with the dimension of a mass and hence give rise to an invariant mass scale in the theory.

Although motivated by phenomenological considerations, the discussion of κ -Poincaré symmetries in four dimensions remains largely heuristic due to the lack of a complete quantum theory of gravity. Moreover, their physical interpretation is complicated by the absence of a derivation from an action functional and a lack of κ -Poincaré invariant models with non-trivial interactions.

In (2+1) dimensions, the situation simplifies considerably, both with respect to the quantisation of gravity and to the role of Hopf algebras as quantum symmetries. This is due to the fact that (2+1)-dimensional gravity can be formulated as a Chern-Simons gauge theory in whose quantisation Hopf algebra structures are known to arise. The (2+1)-dimensional case

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has thus served as a toy model for the motivation and interpretation of κ -Poincaré symmetries in higher dimensions.

However, the Hopf algebras which are well-established as a quantum symmetries of (2+1)-gravity are not κ -Poincaré symmetries, but a different set of deformations of the isometry groups of (2+1)-gravity, the Drinfeld doubles $D(U(\mathfrak{su}(1,1)))$, $D(U_q(\mathfrak{su}(1,1)))$, $q \in \mathbb{R}$, and $D(U_q(\mathfrak{su}(1,1)))$, $q \in U(1)$, for, respectively, vanishing, positive and negative cosmological constant. While these Drinfeld doubles are known to arise in the quantisation of the theory [4, 5], the status of κ -Poincaré symmetries is less clear. Previous work investigating their role in (2+1)-gravity focussed solely on the algebra structure while neglecting the coalgebra, which in itself is not sufficient for establishing the presence of a Hopf algebra symmetry². The situation is complicated further by the fact that both types of deformations share certain physical features such as non-commutative position coordinates and group valued momentum variables. Mathematically, both are associated with the factorisation of the isometry groups of (2+1)-gravity into the (2+1)-dimensional Lorentz group $SU(1,1)$ and the group $AN(2)$.

The purpose of this article is to review the status of Drinfeld doubles and κ -Poincaré symmetries as symmetries of quantised (2+1)-gravity. We explain how the presence or absence of these Hopf algebras in quantised (2+1)-gravity can be determined by considering their classical limit in the Chern-Simons formulation of (2+1)-gravity [6]. It is well known that the Hopf algebras arising in the quantisation of Chern-Simons theory are the quantum counterparts of certain Poisson-Lie structures which describe the Poisson structure on the phase space of the classical theory. By considering the classical limit, one is thus able to reduce the question about the presence of these Hopf algebra symmetries to the analogous question for the associated classical and infinitesimal structures, Poisson-Lie groups and Lie bialgebras. Using the results of [6, 7], we address this question for the Poisson-Lie structures and Lie bialgebras associated with the Drinfeld doubles and a generalised version of the κ -Poincaré symmetries.

The paper is structured as follows: In Sect. 2 we summarise the relevant facts on Hopf algebra symmetries, Poisson-Lie groups and Lie bialgebras. Sect. 3 gives a brief overview of Chern-Simons theory with a focus on the Chern-Simons formulation of (2+1)-gravity. In Sect. 4, we discuss the role of Poisson-Lie structures in the description of the phase space of the theory and explain how one can determine if a given Poisson-Lie structure and the associated Hopf algebra are compatible with a given Chern-Simons theory. In Sect. 5, we apply this procedure to the Poisson-Lie structures associated with the Drinfeld doubles and a generalised version of κ -Poincaré symmetries, whose role in (2+1)-dimensional gravity has been investigated in, respectively, [7] and [6]. Sect. 6 contains our outlook and conclusions.

²It can be shown that both the Drinfeld double $D(U(\mathfrak{su}(1,1)))$ and the κ -Poincaré algebra are isomorphic to the (2+1)-dimensional Poincaré algebra as *algebras*. Hence, the algebra structure alone does not allow one to distinguish these Hopf algebras from each other and from the (2+1)-dimensional Poincaré algebra.

2 κ -Poincaré and quantum double symmetries as Hopf algebra symmetries

Both κ -Poincaré symmetries and Drinfeld doubles are examples of Hopf algebra symmetries which generalise the usual notion of group symmetries in physics. A Hopf algebra is an associative algebra \mathfrak{G} with additional structures, the coproduct $\Delta : \mathfrak{G} \rightarrow \mathfrak{G} \otimes \mathfrak{G}$, the counit $\epsilon : \mathfrak{G} \rightarrow \mathbb{C}$ and the antipode $S : \mathfrak{G} \rightarrow \mathfrak{G}$ which satisfy certain consistency conditions and compatibility conditions with the algebra structure. The role of the coproduct is that it allows one to define representations of \mathfrak{G} on the tensor product of representation spaces of \mathfrak{G} , which can be thought of as multi-particle states. The antipode gives rise to representations of \mathfrak{G} on the duals of its representation spaces, which are usually interpreted as anti-particles. Group symmetries in quantum systems present a trivial example of Hopf algebra symmetries, in which the Hopf algebra is the universal enveloping algebra of the associated Lie algebra $\mathfrak{G} = U(\mathfrak{g})$. Coproduct and antipode take the form $\Delta(\mathbf{x}) = 1 \otimes \mathbf{x} = \mathbf{x} \otimes 1$, $S(\mathbf{x}) = -\mathbf{x} \forall \mathbf{x} \in \mathfrak{g}$, and one recovers the familiar formulas for the symmetries associated with multi-particle states and antiparticle systems.

Hopf algebras can be viewed as the quantum counterparts of corresponding classical structures, namely Poisson-Lie groups. A Poisson-Lie group is a Lie group G equipped with a Poisson structure such that the multiplication $G \times G \rightarrow G$ is a Poisson map. For the trivial case of universal enveloping algebras discussed above, this Poisson structure is the trivial one. Poisson-Lie group structures are in turn defined by associated infinitesimal structures, Lie bialgebras. A Lie bialgebra \mathfrak{g} is a Lie algebra with an additional structure, the cocommutator $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ that satisfies certain compatibility condition with the Lie bracket. It can be viewed as the infinitesimal counterpart or first-order term of the coproduct $\Delta : \mathfrak{G} \rightarrow \mathfrak{G} \otimes \mathfrak{G}$, while the Lie bracket corresponds to the first order term of the commutator of \mathfrak{G} .

For quasi-triangular Hopf algebras such as the Drinfeld doubles and κ -Poincaré symmetries, the corresponding Lie bialgebra structures are in turn defined by an additional structure, the classical r -matrix for the Lie algebra \mathfrak{g} . This is an element $r = r^{\alpha\beta} X_\alpha \otimes X_\beta \in \mathfrak{g} \otimes \mathfrak{g}$ which satisfies the classical Yang Baxter equation (CYBE)

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad (2.1)$$

where $r_{12} = r^{\alpha\beta} X_\alpha \otimes X_\beta \otimes 1$, $r_{13} = r^{\alpha\beta} X_\alpha \otimes 1 \otimes X_\beta$, $r_{23} = r^{\alpha\beta} 1 \otimes X_\alpha \otimes X_\beta$ and X_α , $\alpha = 1, \dots, \dim \mathfrak{g}$, is a basis of \mathfrak{g} .

The relationship between Hopf algebras, Poisson-Lie groups, Lie bialgebras and classical r -matrices is illustrated in Table 1: Via the identity

$$\delta(X_\gamma) = (1 \otimes \text{ad}_{X_\gamma} + \text{ad}_{X_\gamma} \otimes 1)(r) = r^{\alpha\beta} (X_\alpha \otimes [X_\gamma, X_\beta] + [X_\gamma, X_\alpha] \otimes X_\beta) \quad (2.2)$$

the classical r -matrix defines the cocommutator of \mathfrak{g} and hence its Lie bialgebra structure. Exponentiation then yields the associated Poisson-Lie group and quantisation the corresponding Hopf algebra.

An important notion is the concept of Hopf algebra duality. Both the κ -Poincaré and the Drinfeld double symmetries considered in this paper correspond to a pair of Hopf algebras

in duality. For the κ -Poincaré symmetries these are the κ -Poincaré algebra and its dual, the κ -Poincaré group [8]. In the case of Drinfeld double symmetries, one has the Drinfeld double and its dual. Two Hopf algebras are in duality if they are dual as vector spaces, their antipodes are in duality and the comultiplication and counit of one of them are dual to, respectively, the multiplication and unit of the other. In the quasi-triangular case, the Poisson-Lie structures corresponding to such a pair of dual Hopf algebras are a pair of Poisson-Lie groups in duality, the Sklyanin Poisson structure and its dual, the dual Poisson structure. The infinitesimal counterpart of such a pair of Poisson-Lie groups in duality is a pair of dual Lie bialgebras, i. e. two Lie algebras which are dual as vector spaces and such that the Lie bracket of one is dual to the cocommutator of the other and vice versa. These notions of duality are illustrated in Table 1.

For the purpose of this paper, it is important to note that the classical r -matrix defines all of the structures introduced above. This implies that a quasi-triangular Hopf algebra is essentially determined by the corresponding classical r -matrix and allows one to reduce the question about the role of Hopf algebras as symmetries of quantised (2+1)-gravity to a question about the compatibility of the associated classical r -matrices with the action functional defining the theory.

Hopf algebras	Hopf algebra \mathfrak{G} κ -Poincaré algebra Drinfeld doubles	←Hopf algebra duality→	dual Hopf algebra \mathfrak{G}^* κ -Poincaré group dual of Drinfeld doubles
	↑ quantisation: $\{ , \} \rightarrow i\hbar[,], \delta \rightarrow \Delta$		
classical counterparts: Poisson-Lie groups	dual Poisson structure on G	←Poisson-Lie group duality→	Sklyanin bracket on G
	↑ exponentiation		
infinitesimal counterparts: Lie bialgebras	Lie bialgebra \mathfrak{g} $[X_\alpha, X_\beta] = f_{\alpha\beta}^\gamma X_\gamma$ $\delta(X_\alpha) = g_\alpha^{\beta\gamma} X_\beta \otimes X_\gamma$	←duality of Lie bialgebras→	dual Lie bialgebra \mathfrak{g}^* $\delta_*(X_*^\alpha) = f_{\beta\gamma}^\alpha X_*^\beta \otimes X_*^\gamma$ $[X_*^\alpha, X_*^\beta]_* = g_\gamma^{\alpha\beta} X_*^\gamma$
	↑ cocommutator $\delta(X_\alpha) = (1 \otimes \text{ad}_{X_\alpha} + \text{ad}_{X_\alpha} \otimes 1)(r)$		
classical r -matrix	$r = r^{\alpha\beta} X_\alpha \otimes X_\beta \in \mathfrak{g} \otimes \mathfrak{g}$		

Table 1: Hopf algebras, Poisson-Lie groups and Lie bialgebras

3 The Chern-Simons formulation of (2+1)-gravity

As shown in [9, 10], (2+1)-dimensional gravity can be formulated as a Chern-Simons gauge theory. The two ingredients needed in the definition of a Chern-Simons gauge theory are a Lie group G and a non-degenerate symmetric bilinear form on $\mathfrak{g} = \text{Lie } G$ that is invariant under the adjoint action Ad of G on \mathfrak{g} . Given a gauge group G and an Ad -invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} , the corresponding Chern-Simons theory on a three-manifold M is then given by Chern-Simons action

$$S[A] = \int_M \langle dA \wedge A + \frac{2}{3} A \wedge A \wedge A \rangle, \quad (3.1)$$

where A is the Chern-Simons gauge field which is locally a one-form on M with values in the Lie algebra \mathfrak{g} and transforms under gauge transformations $\gamma : M \rightarrow G$ as $A \mapsto \gamma A \gamma^{-1} + \gamma d\gamma^{-1}$.

In the Chern-Simons formulation of (2+1)-gravity, the gauge groups are the isometry groups of the generic solutions of the theory, three-dimensional Minkowski space, de Sitter space and anti de Sitter space. Adapting the conventions from [10] and denoting by Λ minus the cosmological constant, one finds that these isometry groups are the three-dimensional Poincaré group $ISO(2, 1) = SO(2, 1) \ltimes \mathbb{R}^3$ for $\Lambda = 0$, the group $SO(2, 1) \times SO(2, 1)$ for $\Lambda > 0$ and the group $PSL(2, \mathbb{C})$ for $\Lambda < 0$ as shown in Table 2.

Λ	spacetime M_Λ	isometry group H_Λ
$= 0$	\mathbb{M}^3	$SO(2, 1) \ltimes \mathbb{R}^3$
> 0	AdS_3	$SO(2, 1) \times SO(2, 1)$
< 0	dS_3	$PSL(2, \mathbb{C})$

Table 2: Generic spacetimes and isometry groups in (2+1)-gravity

The associated Lie algebras $\mathfrak{h}_\Lambda = \text{Lie } H_\Lambda$ take the form

$$[J_a, J_b] = \epsilon_{abc} J^c \quad [J_a, P_b] = \epsilon_{abc} P^c \quad [P_a, P_b] = \Lambda \epsilon_{abc} J^c, \quad (3.2)$$

where J_a , $a = 0, 1, 2$, are the generators of the Lorentz transformations, P_a , $a = 0, 1, 2$, the generators of the translations, ϵ^{abc} is the totally antisymmetric tensor satisfying $\epsilon_{012} = \epsilon^{012} = 1$ and indices are raised and lowered with the three-dimensional Minkowski metric $\eta = \text{diag}(1, -1, -1)$.

The other ingredient in the Chern-Simons formulation of (2+1)-gravity is an Ad -invariant non-degenerate symmetric bilinear form on \mathfrak{h}_Λ . While the choice of this form is unique for semi-simple gauge groups, the space of Ad -invariant symmetric bilinear forms is two-dimensional for the Lie algebras \mathfrak{h}_Λ . As shown in [10], for all signs of Λ a basis is given by the forms

$$t(J_a, J_b) = 0, \quad t(P_a, P_b) = 0, \quad t(J_a, P_b) = \eta_{ab}, \quad (3.3)$$

$$s(J_a, J_b) = \eta_{ab}, \quad s(J_a, P_b) = 0, \quad s(P_a, P_b) = \Lambda \eta_{ab}. \quad (3.4)$$

A general Ad-invariant symmetric bilinear form τ on \mathfrak{h}_Λ is a linear combination $\tau = \alpha t + \beta s$, $\alpha, \beta \in \mathbb{R}$. The requirement of non-degeneracy then takes the form $\alpha^2 - \Lambda\beta^2 \neq 0$. Note in particular that, while the form t is non-degenerate for all signs of Λ , the form s becomes degenerate for $\Lambda = 0$.

As shown in [9, 10], the Chern-Simons formulation of (2+1)-gravity is given by the action (3.1) with $G = H_\Lambda$, $\langle \cdot, \cdot \rangle = t$ and is obtained by combining the triad $e^a = e^a_\mu dx^\mu$ and the spin connection $\omega^a = \omega^a_\mu dx^\mu$ in the first order formulation of (2+1)-gravity into a Chern-Simons gauge field

$$A(x) = \omega^a(x)J_a + e^a(x)P_a. \quad (3.5)$$

Using (3.1), (3.5) and (3.3), one can show that the Chern-Simons action (3.1) agrees with the Einstein-Hilbert action for (2+1)-gravity in the first order formulation and that the metric on M is given by $g_{\nu\mu} = \eta_{ab}e^a_\mu e^b_\nu$.

The equations of motion derived from the action (3.1) are a flatness condition on the gauge field A , which combines the requirements of vanishing torsion and constant curvature

$$\begin{aligned} F(A) = dA + \frac{1}{2}[A \wedge A] = 0 \quad \Leftrightarrow \quad T^a = de^a + \epsilon^{abc}\omega_b \wedge e_c = 0 \\ R^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c = -\frac{\Lambda}{2}\epsilon^{abc}e_b \wedge e_c. \end{aligned} \quad (3.6)$$

It can be shown that the equations of motion are independent of the choice of the Ad-invariant symmetric bilinear form on \mathfrak{h}_Λ . However, it is important to stress that different choices of this form nevertheless define physically non-equivalent theories as they affect its Poisson structure and physical interpretation. As we will see in the following that it is precisely the choice of this form that determines if the resulting quantum theory will exhibit κ -Poincaré or Drinfeld double symmetries, understanding how different choices of this form affect the classical theory is crucial for the central aim of this paper.

To clarify the impact of the Ad-invariant, symmetric bilinear form, it is instructive to focus on the extreme cases $\tau = \alpha t$, $\beta = 0$ (the form corresponding to (2+1)-gravity) and $\tau = \beta s$, $\alpha = 0$. Considering a manifold of topology $M \approx \mathbb{R} \times S$ and performing a (2+1)-decomposition of the gauge field, one finds that the Poisson structure for the choice $\tau = \alpha t$ is given by

$$\{e^a_i(x), \omega^b_j(y)\} = \frac{\eta^{ab}}{2\alpha}\epsilon_{ij}\delta_S(x-y) \quad \{e^a_i(x), e^b_j(y)\} = \{\omega^a_i(x), \omega^b_j(y)\} = 0 \quad (3.7)$$

while the other Ad-invariant symmetric bilinear form $\tau = \beta s$ yields

$$\{e^a_i(x), e^b_j(y)\} = \frac{\eta^{ab}}{2\Lambda\beta}\epsilon_{ij}\delta_S(x-y), \quad \{\omega^a_i(x), \omega^b_j(y)\} = \frac{\eta^{ab}}{2\beta}\epsilon_{ij}\delta_S(x-y), \quad \{e^a_i(x), \omega^b_j(y)\} = 0. \quad (3.8)$$

The impact on the physical interpretation of the theory is directly apparent when the theory includes point particles minimally coupled to the Chern-Simons gauge field. As shown in [6], exchanging the Poisson brackets (3.7) and (3.8) amounts to exchanging the internal degrees of freedom of the particles, their mass and internal angular momentum or spin. A similar effect arises when one considers spacetimes with a boundary corresponding to an asymptotic

observer, where switching the (3.7), (3.8) amounts to exchanging the asymptotic degrees of freedom, the total energy and total angular momentum of the universe.

However, the effect is also present for purely topological spacetimes without matter or boundaries [6]. In this case, it can be shown that each closed, non-selfintersecting curve on the spatial surface S is equipped with two canonical observables, which can be viewed as a “mass” or momentum and a “spin” or angular momentum and generalise the associated quantities for particles. Via the Poisson bracket, these observables generate earthquakes (cutting the spatial surface along the curve and rotating the edges of the cut against each other) and grafting (cutting the spatial surface along the curve and inserting a cylinder), which can be viewed as, respectively, a rotation and a translation associated to the curve. Exchanging the Poisson brackets (3.7), (3.8) amounts to switching the two canonical observables with respect to this geometry changing transformations and thus affects their physical interpretation.

4 Hopf algebra symmetries and Poisson-Lie structures in Chern-Simons theory

The advantage of the Chern-Simons formulation of (2+1)-gravity is that it gives rise to a simple and efficient description of the phase space and Poisson structure of the theory. The phase space of Chern-Simons theory with gauge group G on a manifold $M \approx \mathbb{R} \times S_{g,n}$, where $S_{g,n}$ is an oriented two-surface of genus g and with n punctures representing point particles, is the quotient of the space of holonomies along a set of generators of the fundamental group $\pi_1(S_{g,n})$ modulo simultaneous conjugation

$$\mathcal{M}_{g,n}(G) = \text{Hom}(\pi_1(S_{g,n}), G)/G. \quad (4.1)$$

This parametrisation of the phase space establishes a direct relation between its Poisson structure and the theory of Poisson-Lie groups. As shown in [11], the Poisson structure on the phase space $\mathcal{M}_{g,n}(G)$ can be described in terms of an auxiliary Poisson structure on the manifold G^{n+2g} defined in terms of a classical r -matrix for G . Moreover, it has been demonstrated in [12] that the contributions of each puncture and handle of $S_{g,n}$ to this Poisson structure can be decoupled and then correspond, respectively, to a copy of the dual Poisson structure and of the Heisenberg double Poisson structure defined by this classical r -matrix.

The classical r -matrices arising in this description of the phase space are general solutions of the CYBE (2.1) which must satisfy a certain compatibility condition relating them to the Chern-Simons action. This condition states that their symmetric component $r_s = \frac{1}{2}(r^{\alpha\beta} + r^{\beta\alpha})X_\alpha \otimes X_\beta$ is dual to the Ad-invariant symmetric bilinear form in the Chern-Simons action which defines the Poisson structure of the theory.

The Hopf algebra symmetries of the quantised Chern-Simons theory then arise as the quantum counterparts of the Poisson-Lie and Lie bialgebra structures defined by this classical r -matrix. Hence, assuming the existence of a well-defined classical limit, one can use the link between the Poisson structure on phase space and classical r -matrices to determine explicitly if a given quasi-triangular Hopf algebra arises as a quantum symmetry of a given Chern-Simons theory.

For this, one takes the following steps:

1. Determine the associated Poisson-Lie structures and Lie bialgebras via the classical limit.
2. Determine the antisymmetric component $r_a = \frac{1}{2}(r^{\alpha\beta} - r^{\beta\alpha})X_\alpha \otimes X_\beta$ of the classical r -matrix defining these structures via (2.2).
3. Consider the symmetric element $r_s \in \mathfrak{g} \otimes \mathfrak{g}$ corresponding to the Ad-invariant symmetric form in the Chern-Simons action.
4. Determine if the sum $r_s + r_a$ satisfies the CYBE (2.1).

5 Drinfeld doubles and κ -Poincaré symmetries in (2+1)-gravity and Chern-Simons theory

We now apply this procedure to determine the role of Drinfeld double and κ -Poincaré symmetries as quantum symmetries of (2+1)-gravity and, more generally, Chern-Simons theory with gauge group H_Λ . A detailed investigation of the Lie bialgebra and Poisson-Lie group structures underlying the κ -Poincaré symmetries and Drinfeld doubles is given in [13, 14, 7], and the compatibility of κ -Poincaré symmetries with Chern-Simons theories with gauge group H_Λ is investigated in [6].

Using the results of [13, 14, 7], one can determine the antisymmetric component of the classical r -matrix which defines these Lie bialgebra structures via (2.2). As shown in [7], for all values of the cosmological constant, the antisymmetric component of the classical r -matrix defining the Drinfeld doubles is of the form

$$r_a = \frac{1}{2}(P_a \otimes J^a - J_a \otimes P_a) + n^a \epsilon_{abc} J^b \otimes J^c \quad \mathbf{n} \in \mathbb{R}^3. \quad (5.1)$$

For the case of κ -Poincaré symmetries we have from [13, 14]

$$r_a = n^a \epsilon_{abc} (J^b \otimes P^c - P^c \otimes J^b) \quad \mathbf{n} \in \mathbb{R}^3 \quad (5.2)$$

with the standard κ -Poincaré symmetries corresponding to vanishing cosmological constant and a timelike deformation with $\mathbf{n} = (\frac{1}{\kappa}, 0, 0)$. The symmetric element of $\mathfrak{h}_\Lambda \otimes \mathfrak{h}_\Lambda$ dual to a general Ad-invariant bilinear form $\tau = \alpha t + \beta s$ on the Lie algebra \mathfrak{h}_Λ is [6]

$$r_s = \frac{\alpha}{\alpha^2 - \Lambda\beta^2} (P_a \otimes J^a + J_a \otimes P^a) - \frac{\beta}{\alpha^2 - \Lambda\beta^2} (\Lambda J_a \otimes J^a + P_a \otimes P^a). \quad (5.3)$$

Recalling the discussion after (3.4), we note that the normalisation factor $1/\alpha^2 - \Lambda\beta^2$ arising from the duality condition diverges precisely when the form τ becomes degenerate. The requirement that the sum $r_a + r_s$ yields a solution of the CYBE (2.1) then results in conditions on the coefficients α, β in the Ad-invariant symmetric bilinear form and on the vector \mathbf{n} in (5.1), (5.2). For the Drinfeld doubles (5.1) these conditions imply that the solutions of the CYBE must satisfy $\beta = 0$, $\mathbf{n}^2 = -\Lambda/\alpha^2$. The classical r -matrices are then given by

$$r = r_a + r_s = \frac{1}{\alpha} P_a \otimes J^a + n^a \epsilon_{abc} J^b \otimes J^c \quad \mathbf{n}^2 = -\frac{\Lambda}{\alpha^2}. \quad (5.4)$$

The Drinfeld doubles therefore correspond to the Chern-Simons formulation of (2+1)-gravity determined by the Ad-invariant symmetric bilinear form (3.3). The vector \mathbf{n} is timelike, spacelike and lightlike or zero, respectively, for $\Lambda < 0$ (de Sitter case), $\Lambda > 0$ (anti de Sitter case) and $\Lambda = 0$ (Minkowski case). This implies that the Drinfeld doubles are compatible with quantised (2+1)-gravity for all signs of the cosmological constant.

It is shown in [6] that the requirement that $r_s + r_a$ satisfies the CYBE (2.1) yields two sets of solutions for the generalised κ -Poincaré symmetries (5.2). The first is characterised by the conditions $\beta = 0$, $\mathbf{n}^2 = -1/\alpha^2$ and the classical r -matrices

$$r = r_a + r_s = \frac{1}{\alpha}(P_a \otimes J^a + J_a \otimes P^a) + n^a \epsilon_{abc}(J^b \otimes P^c - P^c \otimes J^b) \quad \mathbf{n}^2 = -\frac{1}{\alpha^2}. \quad (5.5)$$

It corresponds again to the Chern-Simons formulation of (2+1)-gravity with Ad-invariant symmetric bilinear form (3.3) and involves a spacelike vector \mathbf{n} for all signs of the cosmological constant.

The other set of solutions is given by the conditions $\alpha = 0$, $\mathbf{n}^2 = -1/\Lambda\beta^2$ and the classical r -matrices

$$r = \frac{1}{\beta}(\Lambda J_a \otimes J^a + P_a \otimes P^a) + n^a \epsilon_{abc}(J^b \otimes P^c - P^c \otimes J^b) \quad \mathbf{n}^2 = -\frac{1}{\Lambda\beta^2} \quad \Lambda \neq 0. \quad (5.6)$$

It exists only for $\Lambda \neq 0$ and corresponds to a Chern-Simons theory with gauge group H_Λ and the non-gravitational Ad-invariant symmetric bilinear form (3.4). It involves a vector \mathbf{n} which is timelike for $\Lambda < 0$ (de Sitter case) and spacelike for $\Lambda > 0$ (anti de Sitter case).

This implies in particular that the usual κ -Poincaré symmetries given by the antisymmetric element (5.2) with a *timelike* vector $\mathbf{n} = (\frac{1}{\kappa}, 0, 0)$ are *not* compatible with the Chern-Simons formulation of (2+1)-gravity. The association of timelike κ -Poincaré symmetries with (2+1)-gravity is possible only in the de Sitter case ($\Lambda < 0$). The corresponding Chern-Simons theory is physically in-equivalent to (2+1)-gravity as it exhibits a different Poisson structure and becomes ill-defined in the limit $\Lambda \rightarrow 0$ in which its Ad-invariant symmetric bilinear form (3.4) degenerates.

6 Outlook and Conclusions

In this article, we clarified the status of κ -Poincaré and Drinfeld doubles as quantum symmetries of (2+1)-gravity and more general Chern-Simons theories based on the local isometry groups of (2+1)-gravity. We showed how the question if a Hopf algebra arises as a quantum symmetry of the theory can be reduced to an explicit condition on the associated classical r -matrices and analysed this condition for the κ -Poincaré and Drinfeld doubles.

As a result, we found that the usual κ -Poincaré symmetries with a timelike vector as a deformation parameter are not compatible with (2+1)-gravity and can be associated with a Chern-Simons theory based on the isometry groups of (2+1)-gravity only in the de Sitter case ($\Lambda < 0$). However, in this case, one does not obtain the Chern-Simons formulation of (2+1)-gravity but a physically in-equivalent Chern-Simons theory based on a different Ad-invariant

bilinear form and with different Poisson structure. This is in contrast to the situation for the Drinfeld doubles and *spacelike* κ -Poincaré symmetries, which are compatible with the Chern-Simons formulation of (2+1)-gravity for all signs of the cosmological constant.

In addition to clarifying the role of κ -Poincaré symmetries as symmetries of quantised (2+1)-gravity and Chern-Simons theory, these results suggest avenues for further research which could be pursued for either the timelike or spacelike version of these symmetries. In both cases, the association of the κ -Poincaré algebras with quantised Chern-Simons theory allows one to define models with κ -Poincaré symmetries on spacetimes with non-trivial topology. Moreover, the possibility of including point particles via minimal coupling to the Chern-Simons gauge field allows one to define non-trivial multi-particle systems with κ -Poincaré symmetries. This provides an explicit relation with an action functional as well as non-trivial particle interactions and therefore could prove useful for the physical interpretation of these symmetries as well as the understanding of their role in quantum gravity.

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